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A problem in generalized magneto-thermoelasticity for an infinitely long annular cylinder

HANY H. SHERIEF* and MAGDY A. EZZAT
Department of Mathematics, Faculty of Science, University of Alexandria, Alexandria, Egypt

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Abstract. The problem of an infinitely long annular cylinder whose inner and outer surfaces are subjected to known surrounding temperatures and are traction-free is considered in the presence of an axial uniform magnetic field. The problem is in the context of generalized magneto-thermoelasticity theory with one relaxation time. The Laplace transform with respect to time is used. A numerical method based on a Fourier-series expansion is used for the inversion process.

Numerical computations for the temperature, displacement and stress distributions as well as for the induced magnetic and electric fields are carried out and represented graphically.

Keywords: elasticity, thermoelasticity, magneto thermoelasticity, heat transfer.

1. Introduction

During the second half of the twentieth century, nonisothermal problems in the theory of elasticity have become increasingly important, This is due mainly to their many applications in widely diverse fields. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses, reducing the strength of the aircraft structure. In the technology of modern propulsive systems, such as jet and rocket engines, the high temperatures associated with combustion processes are the origins of severe thermal stresses. Similar phenomena are encountered in the technologies of space vehicles and missiles, in the mechanics of large steam turbines, and even in shipbuilding, where, strangely enough, ship fractures are often attributed to thermal stresses of moderate intensities [1, p. xi].

Biot [2] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equations for both theories are of the diffusion type predicting infinite speeds of propagation for heat waves contrary to physical observations. Lord and Shulmann [3] introduced the theory of generalized thermoelasticity with one relaxation time by postulating a new law of heat conduction to replace the classical Fourier law. This law contains the heat flux vector as well as its time derivative. It contains also a new constant that acts as a relaxation time. The heat equation of this theory is of the wave-type, ensuring finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motion and the constitutive relations remain the same as those for the coupled and the uncoupled theories. This theory was extended by Dhaliwal and Sherief [4] to general anisotropic media in the presence of heat sources.

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* On leave at Department of Mathematics, University of Qatar, Doha-Qatar.

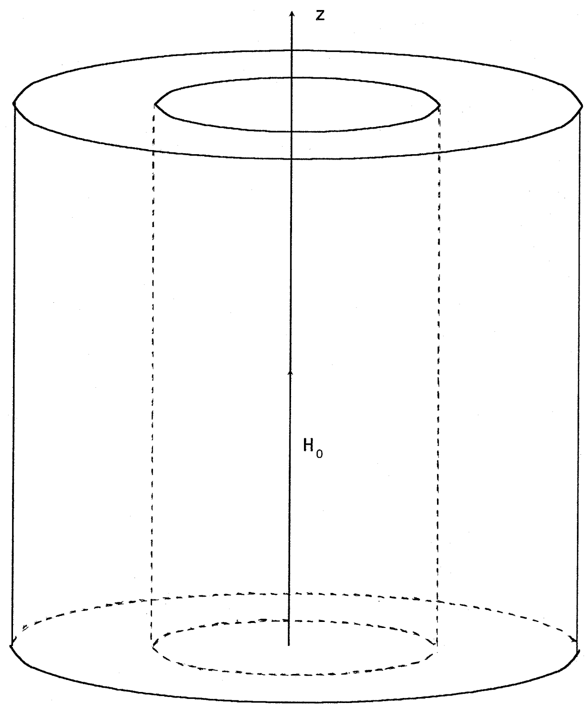

Figure 1. A hollow thermoelastic cylinder with a magnetic field in direction of the axis.

Increasing attention is being devoted to the interaction between magnetic fields and strain in a thermoelastic solid due to its many applications in the fields of geophysics, plasma physics and related topics. In the nuclear field, the extremely high temperatures and temperature gradients, as well as the magnetic fields originating inside nuclear reactors, influence their design and operations [1, p. xi]. Usually, in these investigations the heat equation under consideration is taken as the uncoupled or the coupled equation, not the generalized one. This attitude is justified in many situations, since the solutions obtained from any of these equations differ little quantitatively. However, when short-time effects are considered, the full generalized system of equations has to be used or else a great deal of accuracy is lost [3].

A comprehensive review of the earlier contributions to the subject can be found in [5]. Among the authors who considered the generalized magneto-thermoelastic equations are Nayfeh and Nemat-Nasser [6] who studied the propagation of plane waves in a solid under the influence of an electro-magnetic field. They obtained the governing equations in the general case and the solution for some particular cases. Choudhuri [7] extended these results to rotating media. Lately, Sherief [8] solved a problem for a solid cylinder, while Sherief and Ezzat [9] solved a thermal shock half-space problem using asymptotic expansions.

In this work, the problem of an infinitely long annular cylinder whose inner and outer surfaces are subjected to known surrounding temperatures and are traction-free is considered in the presence of an axial uniform magnetic field in the context of generalized magneto-thermoelasticity theory with one relaxation time. This problem closely models the situation inside some nuclear reactors which are made of elastic materials in the form of annular cylinders and are exchanging heat with the inner and outer mediums through the surfaces of these cylinders.

2. Formulation of the problem

Let (r, ψ, z) be cylindrical polar coordinates with the z-axis coinciding with the axis of an annular infinitely long elastic circular cylinder of a homogeneous, isotropic material of finite conductivity whose inner and outer radii ar R_i, $i = 1, 2$ (see Figure 1). The suffix 1 refers to the inner surface of the cylinder, while the suffix 2 refers to the outer surface. The surfaces of the cylinder are taken to be traction-free and are in contact with media of known temperatures $F_i(r, t)$, $i = 1, 2$. The medium interacts with the surroundings through the heat-transfer coefficients L_i, $i = 1, 2$. A constant magnetic field of strength H_0 acts in the direction of the z-axis. Due to the effect of this magnetic field there arises in the medium an induced magnetic field **h** and an induced electric field **E** (both assumed to be small). Also, there arises a force **F** (the Lorentz Force). Due to the effect of the force, points of the medium undergo a displacement **u** which gives rise to a temperature. The electromagnetic quantities satisfy Maxwell's equations

$$\operatorname{curl}\mathbf{h} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{1}$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\operatorname{div}\mathbf{h} = 0, \qquad \operatorname{div}\mathbf{E} = 0, \tag{3}$$

$$\mathbf{B} = \mu_0(\mathbf{H}_0 + \mathbf{h}), \qquad \mathbf{D} = \varepsilon_0\mathbf{E}. \tag{4}$$

where **J** is the electric current density, μ_0 and ε_0 are the magnetic and electric permeabilities, respectively, and **B**, **D** are the magnetic and electric induction vectors, respectively.

The elastic quantities satisfy the equations of motion in vector form

$$\operatorname{div}\sigma + \mathbf{F} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{5}$$

where σ is the stress tensor, **F** the external body force, which is here equal to the Lorentz force and ρ is the density. The last field equation is the equation of energy balance, namely

$$\frac{\partial}{\partial t}[\rho c_E T + \gamma T_0 e] = -\operatorname{div}\mathbf{q}, \tag{6}$$

where **q** is the heat-flux vector, c_E is the specific heat at constant strain, $e = \operatorname{div}\mathbf{u}$ is the cubical dilatation, γ is a material constant equal to $(3\lambda + 2\mu)\alpha_t$, where λ, μ are Lame's modulii and α_t is the coefficient of linear thermal expansion. T_0 is a reference temperature assumed to be such that $|(T - T_0)/T_0| \ll 1$.

The above field equations are supplemented by constitutive equations which consist first of Ohm's law for moving media [1, p. 726]

$$\mathbf{J} = \sigma_0\left[\mathbf{E} + \mu_0\frac{\partial \mathbf{u}}{\partial t} \times (\mathbf{H}_0 + \mathbf{h})\right],$$

where σ_0 is the electric conductivity. We may linearize the above equation by neglecting small quantities of the second order giving

$$\mathbf{J} = \sigma_0\left[\mathbf{E} + \mu_0\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H}_0\right]. \tag{7}$$

The second constitutive equation is the one for the Lorentz force which is [1, p. 702]

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{8}$$

The third constitutive equation is the Hooke–Duhamel–Neumann law, namely [3]

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e \delta_{ij} - \gamma(T - T_0)\delta_{ij}, \tag{9}$$

where δ_{ij} is Kronecker's delta tensor and e_{ij} is the strain tensor whose components are given by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,1}) \tag{10}$$

The last constitutive equation is the generalized Fourier law of heat conduction which has the form

$$\mathbf{q} + \tau_0 \frac{\partial \mathbf{q}}{\partial t} = -k \operatorname{grad} T. \tag{11}$$

Substituting from Equation (9) in Equation (5) and using Equation (10), we arrive at the equations of motion in vector form

$$(\lambda + \mu)\nabla^2 \mathbf{u} + \mu \operatorname{grad} \operatorname{div} \mathbf{u} - \gamma \operatorname{grad} T + \mathbf{F} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{12}$$

Applying the div operator to both sides of Equation (12), we obtain

$$(\lambda + 2\mu)\nabla^2 e - \gamma \nabla^2 T + \operatorname{div} \mathbf{F} = \rho \frac{\partial^2 e}{\partial t^2}, \tag{13}$$

where ∇^2 is Laplace's operator in cylindrical coordinates, given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2} + \frac{\partial^2}{\partial z^2}.$$

In case of dependence on r only, this reduces to

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

Applying the div operator to both sides of Equation (11), then substituting from the resulting equation in Equation (6) and its time derivative, we obtain the generalized heat equation

$$k\nabla^2 T = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)(\rho c_E T + \gamma T_0 e). \tag{14}$$

Because of the cylindrical symmetry of the problem, and if there is no z-dependence of the field variables, all the considered functions will be functions of r and t only. The components of the displacement vector will be taken of the form

$$u_r = u, u_\psi = u_z = 0.$$

The strain tensor components are thus given by

$$e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\psi\psi} = \frac{u}{r}, \qquad e_{zz} = e_{rz} = e_{r\psi} = e_{\psi z} = 0.$$

It follows that the cubical dilatation e is of the form

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} = \frac{1}{r}\frac{\partial (ru)}{\partial r}. \tag{15}$$

From Equation (9) we obtain the components of the stress tensor as

$$\sigma_{rr} = 2\mu\frac{\partial u}{\partial r} + \lambda e - \gamma(T - T_0), \tag{16a}$$

$$\sigma_{\psi\psi} = 2\mu\frac{u}{r} + \lambda e - \gamma(T - T_0), \tag{16b}$$

$$\sigma_{zz} = \lambda e - \gamma(T - T_0), \tag{16c}$$

$$\sigma_{rz} = \sigma_{z\psi} = \sigma_{\psi r} = 0. \tag{16d}$$

The induced magnetic field **h** will have one component h in the z-direction, while the induced electric field **E** will have one component E in the ψ-direction. From Equation (7), it follows that the electric current density will have one component only in the ψ-direction, given by

$$J = \sigma_0\left[E - \mu_0 H_0\frac{\partial u}{\partial t}\right]. \tag{17}$$

The vector Equations (1) and (2), reduce to the following scalar equations

$$\frac{\partial h}{\partial r} = -\left[J + \varepsilon_0\frac{\partial E}{\partial t}\right], \tag{18}$$

$$\frac{1}{r}\frac{\partial (rE)}{\partial r} = -\mu_0\frac{\partial h}{\partial t}. \tag{19}$$

Eliminating J between Equation (17) and (18), we obtain

$$\frac{\partial h}{\partial r} = \sigma_0\mu_0 H_0\frac{\partial u}{\partial t} - \left[\sigma_0 E + \varepsilon_0\frac{\partial E}{\partial t}\right]. \tag{20}$$

Eliminating E between Equations (18) and (19), we obtain

$$\left[\nabla^2 - \mu_0\sigma_0\frac{\partial}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right]h = \mu_0\sigma_0 H_0\frac{\partial e}{\partial t}. \tag{21}$$

The Lorentz force has one component F in the r-direction obtained from Equations (8) and (17) as

$$F = -\mu_0 H_0\left[\varepsilon_0\frac{\partial E}{\partial t} + \frac{\partial h}{\partial r}\right]. \tag{22}$$

Substituting from Equation (22) in Equations (13), we obtain upon using Equation (19)

$$(\lambda + 2\mu)\nabla^2 e - \gamma \nabla^2 T + \mu_0^2 \varepsilon_0 H_0 \frac{\partial^2 h}{\partial t^2} - \mu_0 H_0 \nabla^2 h = \rho \frac{\partial^2 e}{\partial t^2}. \tag{23}$$

We shall use the following nondimensional variables

$$r' = c_1 \eta r, \qquad R_i' = c_1 \eta R_i, \qquad u' = g c_1 \eta u, \qquad e' = ge, \qquad \sigma_{ij}' = \frac{g \sigma_{ij}}{\mu}$$

$$t' = c_1^2 \eta t, \qquad \tau_0' = c_1^2 \eta \tau_0, \qquad \theta = \frac{T - T_0}{T_0}, \qquad F_i' = \frac{F_i - T_0}{T_0},$$

$$q' = \frac{q}{k T_0 c_1 \eta}, \qquad L_i' = \frac{L_i}{k c_1 \eta}, \qquad h' = \frac{\eta g}{\sigma_0 \mu_0 H_0} h, \quad E' = \frac{\eta g}{\sigma_0 \mu_0^2 H_0 c_1} E.$$

where $g = \frac{\gamma}{\rho c_E}$, $\eta = \frac{\rho c_E}{k}$; $c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ is the speed of propagation of isothermal elastic waves and L_1, L_2 are the coefficients of heat transfer on the inner and outer surfaces of the cylinder, respectively. In terms of these nondimensional variables, the governing Equations (18), (19), (21), (23) and (14) reduce to (dropping the primes for convenience)

$$\frac{\partial h}{\partial r} = \frac{\partial u}{\partial t} - \left[\nu E + V^2 \frac{\partial E}{\partial t} \right], \tag{24}$$

$$\frac{1}{r} \frac{\partial (rE)}{\partial r} = - \frac{\partial h}{\partial t}, \tag{25}$$

$$\left[\nabla^2 - \nu \frac{\partial}{\partial t} - V^2 \frac{\partial^2}{\partial t^2} \right] h = \frac{\partial e}{\partial t}, \tag{26}$$

$$\nabla^2 e - \varepsilon_1 \nabla^2 \theta - \varepsilon_2 \nu \left(\nabla^2 - V^2 \frac{\partial^2}{\partial t^2} \right) h = \frac{\partial^2 e}{\partial t^2}, \tag{27}$$

$$\nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) (\theta + e), \tag{28}$$

where $1/\nu = \eta/\sigma_0 \mu_0$ is a measure of magnetic viscosity, $V = c_1/c$ where c is the speed of light given by $c^2 = 1/\varepsilon_0 \mu_0$, $\varepsilon_1 = bg/\beta^2$ is the thermoelastic coupling constant where $b = \gamma T_0/\mu$, $\beta^2 = (\lambda + 2\mu)/\mu$ and $\varepsilon_2 = \mu_0 H_0^2/\rho c_1^2$ is the magnetoelastic coupling constant. We note that Equation (15) retains its form.

The nondimensional constitutive equations take the form

$$\sigma_{rr} = \beta^2 e - \frac{2u}{r} - \varepsilon_1 \beta^2 \theta, \tag{29a}$$

$$\sigma_{\psi\psi} = \beta^2 e - 2\frac{\partial u}{\partial r} - \varepsilon_1 \beta^2 \theta, \tag{29b}$$

$$\sigma_{zz} = (\beta^2 - 2)e - \varepsilon_1 \beta^2 \theta, \tag{29c}$$

$$\sigma_{rz} = \sigma_{z\psi} = \sigma_{\psi r} = 0. \tag{29d}$$

The initial conditions of the problem are taken to be homogeneous, while the boundary conditions are taken as follows:

(1) The transverse components of the vector **E** are continuous across the surface of the cylinder, this gives

$$E(R_j, t) = E^j(R_j, t), \quad t > 0, \quad j = 1, 2, \tag{30}$$

where E^1 and E^2 are the components of the electric field intensities in the ψ-direction in free space inside and outside the cylinder, respectively.

(2) The transverse components of the vector **h** are continuous across the surface of the cylinder, this gives

$$h(R_j, t) = h^j(R_j, t), \quad t > 0, \quad j = 1, 2, \tag{31}$$

where h^1 and h^2 are the components of the induced magnetic field in the z-direction in free space inside and outside the cylinder, respectively.

(3) The surfaces of the cylinder are traction-free, *i.e.*

$$\sigma_{rr}(R_j, t) = 0, \quad t > 0, \quad j = 1, 2. \tag{32}$$

(4) The heat-conduction boundary condition

$$q_r(R_1, t) = L_1(F_1 - \theta), \tag{33a}$$

$$q_r(R_2, t) = L_2(\theta - F_2). \tag{33b}$$

In order to utilize Equations (30) and (31) above, we must obtain the induced fields E^j, h^j in the free space surrounding the medium. These quantities satisfy the following nondimensional equations

$$\frac{\partial h^j}{\partial r} = -V^2 \frac{\partial E^j}{\partial t}, \quad j = 1, 2, \tag{34}$$

$$\frac{1}{r}\frac{\partial (rE^j)}{\partial r} = -\frac{\partial h^j}{\partial t}, \quad j = 1, 2. \tag{35}$$

3. Solution in the Laplace-transform domain

Taking the Laplace transform with parameter s (denoted by a bar) of both sides of Equations (24)–(35), we obtain the following set of equations

$$\frac{\partial \bar{h}}{\partial r} = s\bar{u} - (\nu + V^2 s)\bar{E}, \tag{36}$$

$$\frac{1}{r}\frac{\partial (r\bar{E})}{\partial r} = -s\bar{h}, \tag{37}$$

$$[\nabla^2 - \nu s - V^2 s^2]\bar{h} = s\bar{e}, \tag{38}$$

$$(\nabla^2 - s^2)\bar{e} = \varepsilon_1 \nabla^2 \bar{\theta} + \varepsilon_2 \nu (\nabla^2 - V^2 s^2)\bar{h}, \tag{39}$$

$$\nabla^2 \bar{\theta} = (s + \tau_0 s^2)(\bar{\theta} + \bar{e}). \tag{40}$$

The nondimensional constitutive Equations (29) in the Laplace transform domain take the form

$$\bar{\sigma}_{rr} = \beta^2 \bar{e} - \frac{2\bar{u}}{r} - b\bar{\theta}, \tag{41a}$$

$$\bar{\sigma}_{\psi\psi} = \beta^2 \bar{e} - 2\frac{\partial \bar{u}}{\partial r} - b\bar{\theta}, \tag{41b}$$

$$\bar{\sigma}_{zz} = (\beta^2 - 2)\bar{e} - b\bar{\theta}, \tag{41c}$$

The boundary conditions in the Laplace transform domain become

$$\bar{E}(R_j, s) = \bar{E}^j(R_j, s), \quad j = 1, 2. \tag{42}$$

$$\bar{h}(R_j, t) = \bar{h}^j(R_j, t), \quad j = 1, 2. \tag{43}$$

$$\bar{\sigma}_{rr} = 0, \quad j = 1, 2. \tag{44}$$

$$\bar{q}_r(R_1, s) = L_1(\bar{F}_1 - \bar{\theta}). \tag{45a}$$

$$\bar{q}_r(R_2, s) = L_2(\bar{\theta} - \bar{F}_2). \tag{45b}$$

Equations (34) and (35) take the following form in the Laplace-transform domain

$$\frac{\partial \bar{h}^j}{\partial r} = -V^2 s \bar{E}^j, \quad j = 1, 2, \tag{46}$$

$$\frac{1}{r}\frac{\partial (r\bar{E}^j)}{\partial r} = -s\bar{h}^j, \quad j = 1, 2. \tag{47}$$

Eliminating $\bar{h}$ and $\bar{\theta}$ between Equations (38)–(40), we get the following sixth-order differential equation satisfied by $\bar{e}$

$$(\nabla^6 - A\nabla^4 + B\nabla^2 - C)\bar{e} = 0, \tag{48}$$

where

$$A = s[\varepsilon_1(s\tau_0 + 1) + \varepsilon_2\nu + \nu + s(\tau_0 + V^2 + 1) + 1],$$

$$B = s^2\{\varepsilon_1(s\tau_0 + 1)(\nu + sV^2) + \varepsilon_2\nu(s(\tau_0 + V^2) + 1)$$
$$+\nu(s(\tau_0 + 1) + 1) + s(s(\tau_0(V^2 + 1) + V^2) + V^2 + 1)\},$$

$$C = s^4(s\tau_0 + 1)(\varepsilon_2\nu V^2 + \nu + sV^2).$$

It should be noted that the above equations reduce to the usual equations of generalized thermoelasticity without electro-magnetic effects in the limit as ν, V and $\varepsilon_2 \to 0$. Equation (48) can be factorized as

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)\bar{e} = 0, \tag{49}$$

where k_1^2, k_2^2 and k_3^2 are the roots of the characteristic equation

$$k^6 - Ak^4 + Bk^2 - C = 0.$$

These roots are given by

$$k_1^2 = \tfrac{1}{3}[A + 2P \sin Q],$$

$$k_2^2 = \tfrac{1}{3}[A - P \sin Q - \sqrt{3}P \cos Q],$$

$$k_3^2 = \tfrac{1}{3}[A - P \sin Q + \sqrt{3}P \cos Q],$$

where

$$P = \sqrt{A^2 - 3B}, \qquad R = \frac{9AB - 2A^3 - 27C}{2P^3}, \qquad Q = \tfrac{1}{3} \sin^{-1}(R).$$

It is worth noting here that all the above quantities including the roots k_I, being functions of the parameter s of the Laplace transform, are complex in general.

The solution of Equation (49) can be written as the sum

$$\bar{e} = \sum_{i=1}^{3} \bar{e}_i,$$

where $\bar{e}_i$ is the solution of the equation

$$(\nabla^2 - k_i^2)\bar{e}_i = 0. \quad i = 1, 2, 3.$$

Thus, the general solution of Equation (49) has the form

$$\bar{e} = \sum_{i=1}^{3}(A_i I_0(k_i r) + B_i K_0(k_i r)), \tag{50}$$

where A_i and B_i are parameters depending on s only, $i = 1, 2, 3$ and I_0 and K_0 are the modified Bessel functions of order zero of the first and second kinds, respectively.

Eliminating $\bar{h}$, $\bar{e}$ and then $\bar{\theta}$, $\bar{e}$ between Equations (38–40), we find that $\bar{\theta}$ and $\bar{h}$ satisfy the same equation as $\bar{e}$, *i.e.*

$$(\nabla^6 - A\nabla^4 + B\nabla^2 - C)\bar{\theta} = (\nabla^6 - A\nabla^4 + B\nabla^2 - C)\bar{h} = 0$$

$\bar{\theta}$ and $\bar{h}$ are thus given by

$$\bar{\theta} = \sum_{i=1}^{3}(A_i' I_0(k_i r) + B_i' K_0(k_i r)), \tag{51}$$

$$\bar{h} = \sum_{i=1}^{3} (A_i'' I_0(k_i r) + B_i'' K_0(k_i r)), \tag{52}$$

where A_i', B_i', A_i'' and B_i'' are parameters depending on s only. The compatibility between Equations (50–52) and Equations (38) and (40) gives

$$A_i' = \frac{s(1+\tau_o s)}{k_i^2 - s(1+\tau_0 s)} A_i, \qquad B_i' = \frac{s(1+\tau_0 s)}{k_i^2 - s(1+\tau_0 s)} B_i, \tag{53}$$

$$A_i'' = \frac{s}{k_i^2 - \nu s - V^2 s^2} A_i, \qquad B_i'' = \frac{s}{k_i^2 - \nu s - V^2 s^2} B_i. \tag{54}$$

Substituting from Equations (53) and (54) in Equations (51) and (52), we obtain

$$\bar{\theta} = s(1+\tau_0 s) \sum_{i=1}^{3} \frac{(A_i I_0(k_i r) + B_i K_0(k_i r))}{k_i^2 - s(1+\tau_0 s)}, \tag{55}$$

$$\bar{h} = s \sum_{i=1}^{3} \frac{(A_i I_0(k_i r) + B_i K_0(k_i r))}{k_i^2 - \nu s - V^2 s^2}. \tag{56}$$

Substituting from Equation (50) in the Laplace transform of Equation (15) and integrating both sides with respect to r, we obtain

$$\bar{u} = \sum_{i=1}^{3} \frac{1}{k_i} (A_i I_1(k_i r) - B_i K_1(k_i r)). \tag{57}$$

In obtaining Equation (57), we have used the following relations of the Bessel functions [10, p. 142]

$$\int z I_0(z)\,\mathrm{d}z = z I_1(z), \qquad \int z K_0(z)\,\mathrm{d}z = -z K_1(z).$$

Substituting from Equations (56) and (57) in Equation (36), and using the relations [10, p. 138]

$$I_0'(z) = I_1(z), \qquad K_0'(z) = -K_1(z),$$

we obtain

$$\bar{E} = -s^2 \sum_{i=1}^{3} \frac{(A_i I_1(k_i r) - B_i K_1(k_i r))}{k_i (k_i^2 - \nu s - V^2 s^2)}. \tag{58}$$

It should be noted here that the solutions (56) and (58) satisfy Equation (37) identically. Substituting from Equations (50), (55) and (57) in Equation (41a), we obtain

$$\bar{\sigma}_{rr} = \sum_{i=1}^{3} \left\{ \beta^2 \left[1 - \frac{\varepsilon_1 s(1+\tau_0 s)}{k_i^2 - s(1+\tau_0 s)} \right] (A_i I_0(k_i r) + B_i K_0(k_i r)) - \frac{2}{r k_i} (A_i I_1(k_i r) - B_i K_1(k_i r)) \right\}, \tag{59}$$

Differentiating both sides of Equation (57) with respect to r and using the relations [10, p. 139]

$$\frac{\mathrm{d}I_1(z)}{\mathrm{d}z} = I_0(z) - \frac{1}{z}I_1(z), \qquad \frac{\mathrm{d}K_1(z)}{\mathrm{d}z} = -K_0(z) - \frac{1}{z}K_1(z),$$

we obtain

$$\frac{\partial \bar{u}}{\partial r} = \sum_{i=1}^{3}\left[A_i\left(I_0(k_i r) - \frac{1}{k_i r}I_1(k_i r)\right) - B_i\left(K_0(k_i r) + \frac{1}{k_i r}K_1(k_i r)\right)\right]. \tag{60}$$

Substituting from Equations (50), (55) and (60) in Equation (41b), we obtain

$$\begin{aligned}\bar{\sigma}_{\psi\psi} \;=\; &\sum_{i=1}^{3} A_i\left\{\left[\beta^2 - 2 - \frac{\varepsilon_1\beta^2 s(1+\tau_0 s)}{k_i^2 - s(1+\tau_0 s)}\right]I_0(k_i r) + \frac{2}{k_i r}I_1(k_i r)\right\}\\ &+\sum_{i=1}^{3} B_i\left\{\left[\beta^2 + 2 - \frac{\varepsilon_1\beta^2 s(1+\tau_0 s)}{k_i^2 - s(1+\tau_0 s)}\right]K_0(k_i r) + \frac{2}{k_i r}K_1(k_i r)\right\}\end{aligned} \tag{61}$$

In order to obtain the induced field in free space, we eliminate $\bar{E}^j$ between Equations (46) and (47), to obtain

$$(\nabla^2 - V^2 s^2)\bar{h}^j = 0, \quad j = 1, 2, \tag{62}$$

where $\bar{h}^1$ and $\bar{h}^2$ are the solutions of Equation (62) which are bounded at the origin and at infinity, respectively. Thus, we have

$$\bar{h}^1 = A_0 I_0(Vsr), \tag{63}$$

$$\bar{h}^2 = B_0 K_0(Vsr), \tag{64}$$

where A_0 and B_0 are some parameters depending on s only.

Substituting from Equations (63) and (64) in Equation (46), we obtain

$$\bar{E}^1 = -\frac{A_0}{V}I_1(Vsr), \tag{65}$$

$$\bar{E}^2 = \frac{B_0}{V}K_1(Vsr). \tag{66}$$

We shall now use the boundary conditions of the problem to evaluate the unknown parameters of the problem, namely A_i and B_i, $i = 0, 1, 2, 3$. Equations (30) and (31) in the Laplace-transform domain together with Equations (56), (58), (63), (64), (65) and (66) immediately give

$$s^2\sum_{i=1}^{3}\frac{(A_i I_1(k_i R_1) - B_i K_1(k_i R_1))}{k_i(k_i^2 - \nu s - V^2 s^2)} - \frac{A_0}{V}I_1(VsR_1) = 0, \tag{67}$$

$$s^2\sum_{i=1}^{3}\frac{(A_i I_1(k_i R_2) - B_i K_1(k_i R_2))}{k_i(k_i^2 - \nu s - V^2 s^2)} - \frac{B_0}{V}K_1(VsR_2) = 0, \tag{68}$$

$$s\sum_{i=1}^{3}\frac{(A_iI_0(k_iR_1)+B_iK_0(k_iR_1))}{k_i^2-\nu s-V^2s^2}-A_0I_0(VsR_1)=0, \tag{69}$$

$$s\sum_{i=1}^{3}\frac{(A_iI_0(k_iR_2)+B_iK_0(k_iR_2))}{k_i^2-\nu s-V^2s^2}-B_0K_0(VsR_2)=0. \tag{70}$$

Equations (32) and (59) give

$$\sum_{i=1}^{3}\Bigg\{\beta^2\left[1-\frac{\varepsilon_1 s(1+\tau_0 s)}{k_i^2-s(1+\tau_0 s)}\right](A_iI_0(k_iR_1)+B_iK_0(k_iR_1)) - \frac{2}{R_1k_i}(A_iI_1(k_iR_1)-B_iK_1(k_iR_1))\Bigg\}=0, \tag{71}$$

$$\sum_{i=1}^{3}\Bigg\{\beta^2\left[1-\frac{\varepsilon_1 s(1+\tau_0 s)}{k_i^2-s(1+\tau_0 s)}\right](A_iI_0(k_iR_2)+B_iK_0(k_iR_2)) - \frac{2}{R_2k_i}(A_iI_1(k_iR_2)-B_iK_1(k_iR_2))\Bigg\}=0, \tag{72}$$

In order to use the boundary conditions (33), we shall use the generalized Fourier law of heat conduction (Equation (11)) in nondimensional form, namely

$$q_r+\tau_0\frac{\partial q_r}{\partial t}=-\frac{\partial\theta}{\partial r}.$$

Taking the Laplace transform of the above equation, we obtain

$$\bar{q}_r=-\frac{1}{1+\tau_0 s}\frac{\partial\bar{\theta}}{\partial r}. \tag{73}$$

Using Equation (73), we may write Equations (33) as

$$\frac{\partial\bar{\theta}}{\partial r}-L_1(1+\tau_0 s)\bar{\theta}=-L_1(1+\tau_0 s)\bar{F}_1,\quad \text{at } r=R_1, \tag{74}$$

$$\frac{\partial\bar{\theta}}{\partial r}+L_2(1+\tau_0 s)\bar{\theta}=L_1(1+\tau_0 s)\bar{F}_2,\quad \text{at } r=R_2, \tag{75}$$

Using Equations (74) and (75) together with Equation (55), we obtain

$$s\sum_{i=1}^{3}\frac{A_i[k_iI_1(k_iR_1)-L_1(1+\tau_0 s)I_0(k_iR_1)] - B_i[k_iK_1(k_iR_1)+L_1(1+\tau_0 s)K_0(k_iR_1)]}{k_i^2-s(1+\tau_0 s)}=-L_1\bar{F}_1, \tag{76}$$

$$s\sum_{i=1}^{3}\frac{A_i[k_i I_1(k_i R_2)+L_1(1+\tau_0 s)I_0(k_i R_2)]}{}$$
$$\frac{-B_i[k_i K_1(k_i R_2)-L_1(1+\tau_0 s)K_0(k_i R_2)]}{k_i^2-s(1+\tau_0 s)}=L_2\bar{F}_2, \tag{77}$$

Equations (67–72), (76) and (77) constitute a system of eight linear equations in the eight unknown parameters $A_i, B_i, i = 0,1,2,3$. Whose solution completes the solution of the problem in the Laplace transform domain. During the numerical inversion of the Laplace transform, these equations are solved numerically.

4. Inversion of the Laplace transfrom

We shall now outline the numerical inversion method used to find the solution in the physical domain. This numerical technique has the advantage that it is easy to implement (relatively speaking), gives good results, and converges quickly. Let $\bar{f}(r,s)$ be the Laplace transform of a function $f(r,t)$. The inversion formula for Laplace transforms can be written as

$$f(r,t)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\bar{f}(r,s)\,\mathrm{d}s$$

where d is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}(r,s)$. Taking $s = d+iy$, we see that the above integral takes the form

$$f(r,t)=\frac{e^{\mathrm{d}t}}{2\pi}\int_{-\infty}^{\infty} e^{ity}\bar{f}(r,d+iy)\,\mathrm{d}y.$$

Expanding the function $h(r,t) = \exp(-\mathrm{d}t)f(r,t)$ in a Fourier series in the interval $[0,2T]$, we obtain the approximate formula [11]

$$f(r,t)=f_\infty(r,t)+E_D,$$

where

$$f_\infty(r,t)=\tfrac{1}{2}c_0(r,t)+\sum_{k=1}^{\infty}c_k(r,t)\quad\text{for } 0\leqslant t\leqslant 2T, \tag{78}$$

and

$$c_k(r,t)=\frac{e^{\mathrm{d}t}}{T}\operatorname{Re}[e^{ik\pi t/T}\bar{f}(r,d+ik\pi/T)],\quad k=0,1,2, \tag{79}$$

where E_D, the discretization error, can be made arbitrarily small if d is taken large enough [11].

Since the infinite series in Equation (78) can only be summed up to a finite number of N terms, the approximate value of $f(r,t)$ becomes

$$f_N(r,t)=\tfrac{1}{2}c_0(r,t)+\sum_{k=1}^{N}c_k(r,t)\quad\text{for } 0\leqslant t\leqslant 2T. \tag{80}$$

Using the above formula to evaluate $f(r,t)$, we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the 'Korrecktur' method [11] is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and hence to accelerate convergence.

The Korrecktur method uses the following formula to evaluate the function $f(r,t)$

$$f(r,t) = f_\infty(r,t) - \mathrm{e}^{2\,\mathrm{d}T} f_\infty(r, 2T+t) + E'_D,$$

where the discretization error $|E'_D| \ll |E_D|$ [11]. Thus, the approximate value of $f(r,t)$ becomes

$$f_{NK}(r,t) = f_N(r,t) - \mathrm{e}^{-2\,\mathrm{d}T} f_{N'}(r, 2T+t), \tag{81}$$

where N' is an integer such that $N' < N$.

We shall now decribe the ε-algorithm that is used to accelerate the convergence of the series in Equation (80). Let N be an odd natural number, and let

$$s_m(r,t) = \sum_{k=1}^{m} c_k(r,t)$$

be the sequence of partial sums of (80). We define the ε-sequence by

$$\varepsilon_{0,m} = 0, \varepsilon_{1,m} = s_m$$

and

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + 1/(\varepsilon_{p,m+1} - \varepsilon_{p,m}), \quad p = 1, 2, 3, \ldots .$$

It can be shown that [11] the sequence

$$\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots, \varepsilon_{N,1}$$

converges to $f(r,t) + E_D - c_0/2$ faster than the sequence of partial sums

$$s_m, \quad m = 1, 2, 3, \ldots .$$

The actual procedure used to invert the Laplace transforms consists of using Equation (81) together with the ε-algorithm. The values of d and T are chosen according to the criteria outlined in [11].

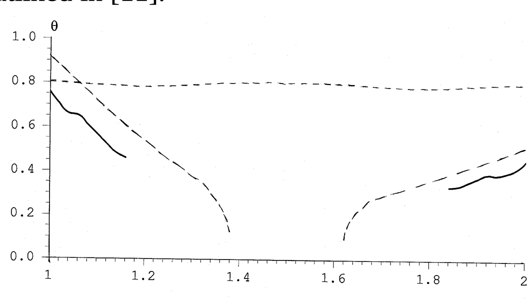

Figure 2. Temperature distribution.

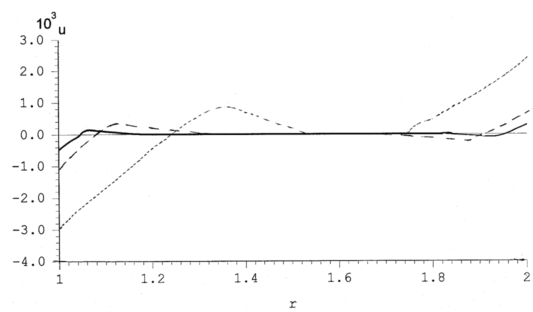

Figure 3. Displacement distribution.

Table 1. Values of the constants

$k = 386$	$\alpha_t = 1{\cdot}78(10)^{-5}$	$c_E = 383{\cdot}1$	$\eta = 8886{\cdot}73$
$\mu = 3{\cdot}86(10)^{10}$	$\lambda = 7{\cdot}76(10)^{10}$	$\rho = 8954$	$c_1 = 4{\cdot}158(10)^{3}$
$\beta^2 = 4$	$\mu_0 = 4\pi(10)^{-7}$	$\varepsilon_0 = (10)^{-9}/(36\pi)$	$\sigma_0 = 5{\cdot}7(10)^{7}$
$V = 1{\cdot}39(10)^{-5}$	$T_0 = 293$	$b = 0{\cdot}042$	$g = 1{\cdot}61$
$\varepsilon_1 = 0{\cdot}0168$	$\varepsilon_2 = 0{\cdot}0008$	$\nu = 0{\cdot}008$	$\tau_0 = 0{\cdot}02$
$R_1 = 1$	$R_2 = 2$	$L_1 = L_2 = 2$	$H_0 = 1$

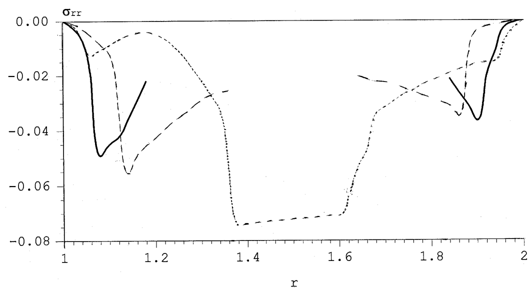

Figure 4. Radial stress distribution.

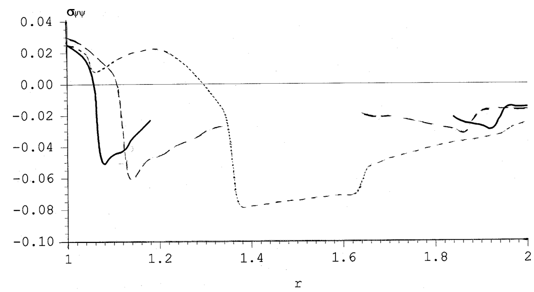

Figure 5. Transverse stress distribution.

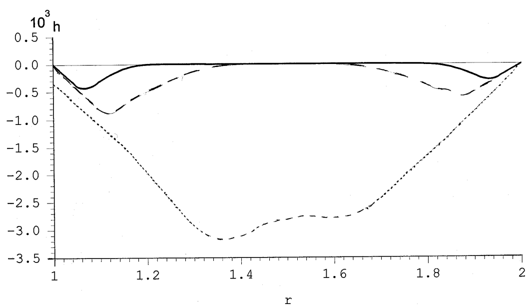

Figure 6. Induced magnetic field distribution.

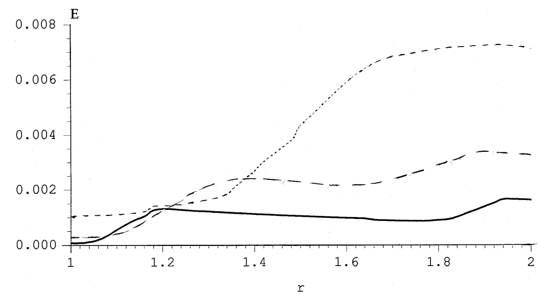

Figure 7. Induced elctric field distribution.

5. Numerical results

The copper material was chosen for purposes of numerical evaluations. The constants of the problem are shown in Table 1

The normalized temperatures of the inner and outer surroundings were taken, respectively, as

$$F_1(t) = 1{\cdot}5 + 0{\cdot}5\ \cos(13t), \qquad F_2(t) = 1.$$

The computations were performed for three values of nondimensional time, namely $t = 0{\cdot}06$, $t = 0{\cdot}12$ and $t = 0{\cdot}36$. The numerical technique outlined above was used to obtain the temperature, displacement, radial stress and transverse stress distributions, as well as the induced magnetic and electric field distributions. In all figures, solid line represent the function when $t = 0{\cdot}06$, the dashed lines represent it when $t = 0{\cdot}12$, while dotted lines represent the function when $t = 0{\cdot}36$. The normalized temperature increment θ is represented by the graph in Figure 2. The displacement u is shown in Figure 3. The stress components σ_{rr} and $\sigma_{\psi\psi}$ are shown in Figures 4 and 5, respectively while the induced fields h and E are shown in Figures 6 and 7, respectively.

The phenomenon of finite speeds of propagation is manifested in all these figures. For the smallest values of time considered we see that the heat effects of the surrounding media are localized in a region adjacent to the walls. This region expands with the passage of time to fill the whole cylinder for the largest value of time. This region corresponds to the propagation of wave fronts from the surfaces of the cylinder. This is not the case when the coupled equation of heat conduction is used. In [12] a similar problem was treated in the context of the coupled theory of thermoelasticity. It is seen there that the thermal effects extends to the whole cylinder immediately.

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